SPRING 2021: MATH 147 QUIZ 9 SOLUTIONS

You must show all work to receive full credit. Each problem is worth 5 points.

1. Set up the integral that calculates the arc length of the curve $C : \mathbf{r}(t) = (t^2+1, 4t^3+3)$, $-1 \le t \le 0$. Do not evaluate this integral.

Solution. $\mathbf{r}'(t) = (2t, 12t^2)$, so $||\mathbf{r}'(t)|| = \sqrt{(2t)^2 + (12t^2)^2} = \sqrt{4t^2 + 144t^4}$. Thus,

$$\text{arc length}(C) = \int_{-1}^{0} \sqrt{4t^2+144t^4}\ dt.$$

Note: In calculating the integral above, it is tempting to write

$$\sqrt{4t^2+144t^4} = 2t\sqrt{1+36t^2},$$

which is not correct. Actually, $\sqrt{4t^2+144t^4} = |2t|\sqrt{1+36t^2}$. Since $-1 \le t \le 0$, $|2t| = -2t$. Thus,

$$\int_{-1}^{0} \sqrt{4t^2+144t^4}\ dt = \int_{-1}^{0} -2t\sqrt{1+36t^2}\ dt,$$

which can then be calculated using u-substitution.

2. Set up the line integral $\int_C f(x,y,z)\ ds$, for $f(x,y,z) = x+y+yx$ and $C : \mathbf{r} = (\sin(t), \cos(t), t)$, with $0 \le t \le 2\pi$, as a single integral using the given parametrization of C. Do not evaluate the resulting single integral.

Solution. $\mathbf{r}'(t) = (\cos(t), -\sin(t), 1)$, so $||\mathbf{r}'(t)|| = \sqrt{(\cos(t))^2 + (-\sin(t))^2 + 1^2} = \sqrt{2}$. Thus,

$$\int_C x+y+xy\ ds = \int_0^{2\pi} \{\sin(t)+\cos(t)+\sin(t)\cos(t)\} \cdot \sqrt{2}\ dt.$$

3. Set up the surface integral $\int\int_S (x^2+y^2)z\ dS$, for S the upper hemisphere of the sphere of radius a centered at the origin, as a double integral over a flat region in $\mathbb{R}^2$. Do not evaluate the the resulting double integral.

Solution. We have that S is given by $G(\rho,\theta) = (a\sin(\phi)\cos(\theta), a\sin(\phi)\sin(\theta), a\cos(\phi))$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$. It follows that

$$\mathbf{T}_\phi \times \mathbf{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos(\phi)\cos(\theta) & a\cos(\phi)\sin(\theta) & -a\sin(\phi) \\ -a\sin(\phi)\sin(\theta) & a\sin(\phi)\cos(\theta) & 0 \end{vmatrix} = a^2\sin(\phi)\cdot\{\sin(\phi)\cos(\theta)\vec{i} + \sin(\phi)\sin(\theta)\vec{j} + \cos(\phi)\vec{k}\}.$$

Since the vector $\sin(\phi)\cos(\theta)\vec{i} + \sin(\phi)\sin(\theta)\vec{j} + \cos(\phi)\vec{k}$ lies on the sphere of radius one, centered at the origin, we have that $||\mathbf{T}_\phi \times \mathbf{T}_\theta|| = |a^2\sin(\phi)| = a^2\sin(\phi)$. Thus,

$$\int\int_S (x^2+y^2)z\ dS = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} [(a\sin(\phi)\cos(\theta))^2 + (a\sin(\phi)\sin(\theta))^2] a\cos(\phi) \cdot a^2\sin(\phi)\ d\phi\ d\theta.$$